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Polymer and Fock representations for a Scalar field

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Abstract

In loop quantum gravity, matter fields can have support only on the 'polymer-like' excitations of quantum geometry, and their algebras of observables and Hilbert spaces of states can not refer to a classical, background geometry. Therefore, to adequately handle the matter sector, one has to address two issues already at the kinematic level. First, one has to construct the appropriate background independent operator algebras and Hilbert spaces. Second, to make contact with low energy physics, one has to relate this 'polymer description' of matter fields to the standard Fock description in Minkowski space. While this task has been completed for gauge fields, important gaps remained in the treatment of scalar fields. The purpose of this letter is to fill these gaps.

I. INTRODUCTION

In this letter, we construct the 'polymer representation' of a real-valued scalar field (such as Klein Gordon) and compare it to the Fock representation in Minkowski space-time. This work is motivated by three considerations.

First, scalar fields provide a simple arena for mathematical investigations of quantum fields in flat and curved space-times. In loop quantum gravity, on the other hand, it is the gauge fields that can be most easily incorporated because of the availability of Wilson loops. For a Maxwell field, for example, not only has the polymer representation been constructed in detail but its relation to the Fock representation is also well-understood [1,2]. To make contact with the rich mathematical quantum field theory literature, it is important to extend these results to the case of a scalar field.

The second motivation comes from the fact that the 'polymer' description of a real-valued scalar field and, in particular, the notion of shadow states used in the semi-classical analysis, is technically more subtle than that in the Maxwell case. As we will see below, there is a

sense in which the gauge group $U(1)$ of the Maxwell theory is now replaced by the additive group of real numbers. Therefore, at the technical level, this analysis provides a first step for incorporation of non-compact gauge groups in the polymer representation.

The third motivation comes from loop quantum gravity itself. In the discussion of the Hamiltonian constraint either in full loop quantum gravity [4] or in the more restricted context of loop quantum cosmology [5], the detailed treatments have focussed only on the σ-model type scalar fields which take values in compact groups. Incorporation of Klein-Gordon type scalar fields requires an extension of the kinematical framework. Conceptually, the existence of such an extension is also needed in the discussion of black hole entropy in quantum geometry, particularly in the proof that the Hawking-Bekenstein formula is correctly [6] modified in presence of a non-minimally coupled scalar field [7].

The letter is organized as follows. In part II, we present the 'polymer description' of a scalar field (see also [3]). Since we wish to compare it with the Fock representation, we will consider scalar fields in Minkowski space. However, this discussion can be trivially generalized to any manifold with topology $\Sigma \times R$ where Σ is any spatial 3-manifold. In part III we recast the Fock representation in a form which facilitates comparison with the polymer description. This construction parallels that in the Maxwell theory [2] which in turn was inspired by key ideas introduced by M. Varadarajan [1]. The comparison is carried out in part IV. Part V presents the overall viewpoint from a quantum geometry perspective.

II. THE POLYMER QUANTUM SCALAR FIELD

As in quantum geometry [8], we will proceed in the following steps: i) Select an algebra of functions on the classical configuration space which are to serve as 'elementary' classical observables; ii) Select a suitable representation of this algebra by operators on a Hilbert space; and, iii) Express momenta as self-adjoint operators on this Hilbert space. Our choices will ensure that the polymer scalar field can 'live' on quantum geometry.

For scalar fields of interest, the classical configuration space $\mathcal{A}$ consists of the space of all real valued, smooth functions ϕ (with an appropriate fall-off) on the spatial 3-plane, R^3. We will first introduce 'holonomy' functions on $\mathcal{A}$ which will constitute the *elementary configuration variables* for the polymer representation. Let us begin with some definitions. We will follow the terminology used in the polymer framework for Maxwell (or, more generally, gauge) fields. A set V consisting of a finite number of points on R^3 will be called a *vertex set*. (The empty set is allowed as a vertex set.) Given a vertex set $V = \{x_1, \ldots x_n\}$, denote by Cyl_V the vector space generated by finite linear combinations of the following functions of fields ϕ

$$\mathcal{N}_{V,\vec{\lambda}}(\phi) = e^{i \sum_j \lambda_j \, \phi(x_j)} \tag{2.1}$$

where $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \ldots \lambda_n)$ are arbitrary real numbers. Cyl_V has the structure of a $\star$ algebra. Finally we introduce the space Cyl of *all* cylindrical functions on $\mathcal{A}$:

$$\mathrm{Cyl} = \cup_V \, \mathrm{Cyl}_V$$

We can complete it with respect to the sup norm and obtain a $C^\star$ algebra which we denote by $\overline{\mathrm{Cyl}}$. This can be taken to be *the $C^\star$ algebra of configuration observables.*

Our next task is to find a suitable representation of this algebra. Since $\overline{\rm Cyl}$ is an Abelian $C^\star$-algebra with identity, we can use the Gel'fand theory to conclude that every of its cyclic representations is of the following type: The Hilbert space $\mathcal{H}$ is the space $L^2(\bar{\mathcal{A}}, d\mu)$ of square integrable functions on a compact topological space $\bar{\mathcal{A}}$ with respect to some regular Borel measure μ, and $\overline{\rm Cyl}$ acts on $\mathcal{H}$ by multiplication. $\bar{\mathcal{A}}$ is called the Gel'fand spectrum of the algebra $\overline{\rm Cyl}$.[1] Because $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu)$, it is natural to think of $\bar{\mathcal{A}}$ as the 'quantum configuration space'.

To specify the structure of $\bar{\mathcal{A}}$, let us first consider a vertex set V_o consisting of a single point x_o. Then, ${\rm Cyl}_{V_o}$ is the space of all almost periodic functions on a real line (given by equivalence classes of $\phi(x)$ where two are equivalent if they have the same value at x_o). The Gel'fand spectrum of the corresponding $C^\star$-algebra $\overline{\rm Cyl}_{V_o}$ is the Bohr completion $\bar{R}^{x_o}_{\rm Bohr}$ of the real line R. That is, $\bar{R}^{x_o}_{\rm Bohr}$ is a compact topological space such that $\overline{\rm Cyl}_{V_o}$ is the $C^\star$ algebra of *all* continuous functions on $\bar{R}^{x_o}_{\rm Bohr}$. R is densely embedded in $\bar{R}^{x_o}_{\rm Bohr}$; thus $\bar{R}^{x_o}_{\rm Bohr}$ can be regarded as a completion of R. At the point x_o, whereas the classical fields take values $\phi(x_o)$ in R, the quantum field will take values in $\bar{R}^{x_o}_{\rm Bohr}$. Thus, the classical configuration space $\mathcal{A}$ is now enlarged to a quantum configuration space $\bar{\mathcal{A}}$ consisting of (arbitrarily discontinuous) $\bar{R}_{\rm Bohr}$-valued functions on R^3.

Remark: In the existing treatments of scalar fields in the polymer representation, the λ_j are generally restricted to be integers. Then, the configuration variables are *periodic* functions of $\phi(x_o)$ and the Gel'fand spectrum of $\overline{\rm Cyl}_{V_o}$ is S^1. Physically, this corresponds to considering a $U(1)$ σ-model in Minkowski space where the fields are required to take values in $U(1)$ —rather than R— at each point of the spatial plane R^3. For the Klein-Gordon type real scalar fields, on the other hand, periodic functions do not suffice to separate points of the configuration space $\bar{\mathcal{A}}$. Almost periodic functions are essential and this in turn leads to the Bohr compactification. (For more information on almost periodic functions and the Bohr compactification see for example [9].)

The polymer representation is based on a *preferred, diffeomorphism invariant, faithful Borel measure μ_o on $\bar{\mathcal{A}}$*. This measure is defined by the positive linear functional Γ_o on $\overline{\rm Cyl}$:

$$\Gamma_o(\mathcal{N}_{V,\vec{\lambda}}) = \begin{cases} 1 & \text{if } \lambda_j = 0,\ \forall j \\ 0 & \text{otherwise} \end{cases}$$

Since continuous functions on $\bar{\mathcal{A}}$ are given precisely by the (Gel'fand transform of) elements of $\overline{\rm Cyl}$, to define a regular Borel measure, it suffices to specify values of integrals of all these functions. The measure μ_o defined by Γ_o is given by:

$$\int_{\bar{\mathcal{A}}} d\mu_o\ (\mathcal{N}_{V,\vec{\lambda}}) = \begin{cases} 1 & \text{if } \lambda_j = 0,\ \forall j \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

The quantum Hilbert space of the polymer representation is $\mathcal{H}_{\rm poly} = L^2(\bar{\mathcal{A}}, d\mu_o)$. With each

[1] In what follows only the following facts about $\bar{\mathcal{A}}$ are important: i) $\bar{\mathcal{A}}$ is a compact topological space, the $C^\star$-algebra of *all* continuous functions on which is naturally isomorphic with $\overline{\rm Cyl}$. (For notational simplicity, we will identify the two $C^\star$ algebras); and, ii) There is a natural, dense embedding of the classical configuration space $\mathcal{A}$ into $\bar{\mathcal{A}}$.

pair (x_o, λ_o), there is an 'elementary' configuration/holonomy operator $\hat{h}(x, \lambda)$ which acts by multiplication: For all cylindrical functions $\Psi(\phi)$ we have

$$\hat{h}(x_o, \lambda_o)\, \Psi(\phi) = e^{i\lambda_o \phi(x_o)}\, \Psi(\phi) \tag{2.3}$$

These operators are unitary. But because the positive linear functional Γ is discontinuous in λ, $h(x, \lambda)$ fail to be weakly continuous in λ whence there is no operator $\hat{\phi}(x)$ on $\mathcal{H}_{\rm poly}$. This is completely analogous to what happens in the polymer representation of the Maxwell field where the holonomy operators $\hat{h}_e$ are well defined for every edge e but the connection operator $\hat{A}$ is not.

Finally, we turn to the momentum operators. For the classical scalar field, with each test function g on R^3 one associates a momentum functional

$$P(g) = \int_{R^3} d^3x\ \pi g$$

where π is the momentum conjugate to ϕ. Note that this definition does not require the volume form on R^3 since π by itself is a density of weight one. The Poisson brackets between these momenta and the holonomy functionals are given by:

$$\{P(g), h(x_o, \lambda_o)\} = i\lambda_o\, g(x_o) h(x_o, \lambda_o) \tag{2.4}$$

To implement these relations in the quantum theory, for each test function g on R^3 we define a momentum operator $\hat{P}(g)$ whose action on Cyl is given by:

$$[\hat{P}(g)\, \mathcal{N}_{V,\vec{\lambda}}](\phi) = \left(\hbar \sum_j \lambda_j g(x_j)\right)\ \mathcal{N}_{V,\vec{\lambda}}(\phi) \tag{2.5}$$

These operators are essentially self-adjoint on $\mathcal{H}_{\rm poly}$. They are the analogs of the smeared electric field operators in the Maxwell case. Their eigenvectors are simply $\mathcal{N}_{V,\vec{\lambda}}$. In the Maxwell case, since the gauge group is $U(1)$ rather than R, in place of the real numbers λ_j, we had *integers* n_j; the present λ_j are the continuous analogs of 'fluxons' (or 'spins' of quantum geometry). As in quantum Maxwell theory or quantum geometry, we can decompose $\mathcal{H}_{\rm poly}$ as a direct sum of Hilbert spaces associated with vertex sets:

$$\mathcal{H}_{\rm poly} = \bigoplus_V \mathcal{H}_V$$

where the direct sum is over arbitrary vertex sets $V \equiv \{x_1, \ldots x_n\}$ (including the empty vertex set) and each $\mathcal{H}_V$ is the Cauchy completion of finite linear combinations of $\mathcal{N}_{V,\vec{\lambda}}$, where each λ_j is *non-zero*. (If a λ_j vanishes, that function does not depend on $\phi(x_j)$ and thus appears in the Hilbert space $\mathcal{H}'_\beta$ associated with a vertex set β with one less point, x_j.) Thus, an orthonormal basis in $\mathcal{H}_{\rm poly}$ is given by the functions $\mathcal{N}_{V,\vec{\lambda}}(\phi)$, where $\vec{\lambda} \equiv \{\lambda_1, \ldots \lambda_N\}$ are *non-zero* real numbers. Following the terminology in quantum geometry, we will refer to these basis vectors as *scalar network functions*.

Next, let us note the commutation relations between the 'holonomy' and the momentum operators in the polymer representation. The 'holonomy' operators commute among themselves and so do the momentum operators. The only non-trivial commutators are

$$[\hat{P}(g),\, \hat{h}(x_o, \lambda_o)] \,=\, \hbar\lambda_o\, g(x_o)\; \hat{h}(x_o, \lambda_o) \tag{2.6}$$

These commutation relations implement the Poisson relations (2.4) among the classical variables and will be used in section III. Note that the representation of the holonomy-momentum algebra we have thus obtained is *irreducible* and was constructed without *any* reference to a background field such as a metric; the measure μ_o is diffeomorphically invariant and all our constructions are diffeomorphically covariant.

Remark: While we worked in the continuum to facilitate comparison with the Fock representation, our construction is motivated by the requirement that the polymer description of the scalar field be well-defined also on a quantum geometry [8]. In this case, the fundamental excitations of geometry are 1-dimensional, polymer-like, and R^3 is replaced by arbitrary graphs (which can be regarded as 'floating lattices'). Geometry and gauge fields have support on these graphs. Scalar fields on the other hand will reside only at vertices. Thus, by identifying the 'vertex-sets' of this section with the set of vertices of the standard graphs of quantum geometry, we can do physics of the quantum scalar field on quantum geometry [10].

III. THE r-FOCK DESCRIPTION

The standard Fock description can be cast in the following convenient form. The Hilbert space $\mathcal{H}_{\rm Fock}$ is $L^2(\mathcal{S}^\star, d\mu_F)$, where $\mathcal{S}^\star$ the space of tempered distributions on R^3 and $d\mu_F$ is the Gaussian Fock measure on it. The basic operators can be taken to be $\hat{U}(f) = \exp i \int d^3x\, \hat{\phi}(x) f(x)$ and $\hat{\pi}(g) = \int d^3x\, \hat{\pi}(x) g(x)$, where f and g are test functions on R^3. The first act by multiplication:

$$[\hat{U}(f)\,\Psi](\tilde{\phi}) \,=\, e^{i\int d^3x \tilde{\phi}(x) f(x)}\;\; \Psi(\tilde{\phi})\,, \tag{3.1}$$

where $\tilde{\phi} \in \mathcal{S}^\star$. These operators are unitary, and $\hat{U}(\lambda f)$ are weakly continuous in the real parameter λ, whence $\int d^3x \hat{\phi}(x) f(x)$ exist as self-adjoint operators. The momentum operators act via derivation:

$$[\hat{\pi}(g)\,\Psi](\tilde{\phi}) \,=\, \frac{\hbar}{i}[\, \int d^3x\, g(x)\, \frac{\delta}{\delta\tilde{\phi}(x)} \,-\, \int d^3x\, g(x)\, \Delta^{\frac{1}{2}}\, \tilde{\phi}(x)\,]\, \Psi(\tilde{\phi}) \tag{3.2}$$

where the second term arises because the 'divergence' of the Gaussian Fock measure with respect to the vector field $\int d^3x\, g(x) \delta/\delta\phi(x)$ is non-zero.

We will now construct an isomorphic description in which Fock states are represented as *square-integrable functions on $\bar{\mathcal{A}}$ with respect to a new measure $\mu_F^{(r)}$* and discuss the action of operators. This step will enable us to regard both $\mathcal{H}_{\rm poly}$ and $\mathcal{H}^{(r)}_{\rm Fock}$ as Cauchy completions of Cyl (with respect to μ_o and $\mu_F^{(r)}$). Since states in both representations arise as functions on the same space $\bar{\mathcal{A}}$, it will be easier to compare them.

First, for each real number $r > 0$, we define a 'taming map' Λ_r from $\mathcal{S}^\star$ to $\bar{\mathcal{A}}$ as follows. Fix a 2-point smoothening function $f_r(x, y)$ on R^3 such that: i) $f_r(x, y)$ is symmetric in x, y, ii) $f_r(x, y) = g_r(x - y)$ where g_r is a Schwartz test function; and, iii) as a distribution, $f_r(x, y)$ tends to $\delta(x, y)$ as r tends to zero. A concrete example is provided by

$$f_r(x,y) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{e^{-\frac{|x-y|^2}{2r^2}}}{r^3} \tag{3.3}$$

We will often fix x, regard f_r as a function of y and write $f_r(x,y) = f_{r,x}(y)$. Now, given any tempered distribution $\tilde{\phi}$, we will set:

$$[\Lambda_{(r)}\,(\tilde{\phi})](x) = \int d^3y\, f_{r,x}(y)\tilde{\phi}(y)\,. \tag{3.4}$$

The result, $\Lambda_{(r)}\,(\tilde{\phi})$, is a C^∞ function on R^3 and, in particular, defines an element of $\bar{\mathcal{A}}$. It is easy to verify that the linear map,

$$\Lambda_{(r)} : \mathcal{S}^\star \mapsto \bar{\mathcal{A}}\,,$$

is injective. Denote by $\mu_F^{(r)}$ the push-forward of the Fock measure on $\mathcal{S}^\star$. Then, each element of the Fock space can also be represented as a function on $\bar{\mathcal{A}}$ which is square integrable with respect to $\mu_F^{(r)}$. Thus, the Fock space $\mathcal{H}_{\rm Fock}$ is naturally isomorphic with $\mathcal{H}^{(r)}_{\rm Fock} = L^2(\bar{\mathcal{A}}, d\mu_F^{(r)})$. We will call it the r-Fock space. Let us explore the new measure on $\bar{\mathcal{A}}$. Since $\overline{\rm Cyl}$ is the space of *all* continuous functions on $\bar{\mathcal{A}}$, any regular Borel measure is determined by specifying the integrals of functions in Cyl. By linearity, it suffices to calculate the integrals of the 'elementary cylindrical functions' $\mathcal{N}_{V,\vec{\lambda}}(\phi)$. We have:

$$\int_{\bar{\mathcal{A}}} d\mu_F^{(r)}\; \mathcal{N}_{V,\vec{\lambda}}(\phi) \;=\; \int_{\mathcal{S}^\star} d\mu_F\; e^{i\int d^3y \left(\sum_j \lambda_j f_{r,x_j}(y)\right)\tilde{\phi}(y)} = e^{-\frac{1}{4}||\sum_j \lambda_j f_{r,x_j}(y)||^2} \tag{3.5}$$

for all V and $\vec{\lambda}$, where

$$||F||^2 = \int d^3y\, F(y)\Delta^{-\frac{1}{2}}\, F(y).$$

Here, in the last step we have used the well known fact about the standard Fock measure. Thus, we have exhibited the Fourier transform of the measure $d\mu_F^{(r)}$, which characterizes it completely. In particular, each cylindrical function is normalizable and thus determines an element of the r-Fock space. It is interesting —and perhaps even counter-intuitive– that the r-Fock space and the polymer Hilbert space can both be obtained by Cauchy completing Cyl with suitable measures, namely, $d\mu_F^{(r)}$ and $d\mu_o$ respectively.

Next, let us consider the r-images of the basic operators $\hat{U}(f)$ and $\hat{\pi}(g)$. Let us first consider the image of $\hat{U}(\lambda f_{r,x})$ where we have restricted the smearing function f to have the form $f = \lambda f_{r,x}$ for some $\lambda \in R$ and $x \in R^3$. A straightforward calculation then yields the following action of the r-image of this operator on cylindrical functions $\Psi(\phi)$:

$$[\hat{U}_{(r)}(\lambda_o f_{r,x_o})\,\Psi](\phi) = e^{i\lambda_o\phi(x_o)}\,\Psi(\phi) \tag{3.6}$$

Thus, as one might expect, the action is just by multiplication. However, it is interesting that the result is again a cylindrical function and, on the right side, there is no reference to the detailed form of the taming function; only λ_o and x_o appear. Note that since the operator $\hat{U}(\lambda f)$ is weakly continuous in λ on $\mathcal{H}_{\rm Fock}$, the operator $\hat{U}_{(r)}(\lambda f_{r,x_o})$ is weakly continuous in λ on $\mathcal{H}^{(r)}_{\rm Fock}$. Hence, although multiplication by $\phi(x_o)$ does not leave Cyl invariant, it is a

well-defined operation on $\mathcal{H}^{(r)}_{\rm Fock}$. (Recall this was not the case on $\mathcal{H}_{\rm poly}$.) Finally, we have given the explicit formula only when the test functions are of the form $\lambda f_{r,x}$. However, since the vector space generated by these test functions (with arbitrary $\lambda \in R$ and $x \in R^3$) is dense in the space of all test functions, this specification suffices.

Next, let us specify the action of the image of the Fock momentum operators on cylindrical functions. Again, we can first restrict ourselves to test functions of the form $g = \lambda f_{r,x}$ and then extend the action of the operator to arbitrary test functions g. We have:

$$[\hat{\pi}_{(r)}(\lambda_o f_{r,x_o})\mathcal{N}_{V,\vec{\lambda}}](\phi) = \hbar[\int d^3y \lambda_o f_{r,x_o}(y) \sum_j \lambda_j f_{r,x_j}(y) + i\lambda_o(\Delta^{\frac{1}{2}}\,\phi)(x_o)]\mathcal{N}_{V,\vec{\lambda}}(\phi) \qquad (3.7)$$

(Note that $\Delta^{\frac{1}{2}}\phi$ is well-defined for all ϕ in the image of the 'taming map' $\Lambda_{(r)}$, i.e., on the support of the measure $\mu^{(r)}_F$.) Again the second term is the 'divergence' of the vector field '$g(\delta/\delta\tilde{\phi})$' with respect to the measure $\mu^{(r)}_F$. Finally, the only non-trivial commutators between the holonomy and momentum operators are:

$$[\hat{\pi}_{(r)}(g),\, \hat{U}_{(r)}(\lambda_o f_{r,x_o})] = \hbar\lambda_o\,[\int d^3y\, g(y) f_{r,x_o}(y)]\; \hat{U}_{(r)}(\lambda_o f_{r,x_o}) \qquad (3.8)$$

We conclude with two conceptually important remarks.

i) The Fock and the r-Fock representations are naturally *isomorphic*. Therefore, everything one can do in the standard Fock representation —e.g., the introduction of the notion of Hadamard states and regularization of products of operators— can be translated *unambiguously* to the r-Fock representation, i.e. to structures defined on $\bar{\mathcal{A}}$. However, as with the action of the momentum operators, these constructions can involve the operation of multiplication by $\phi(x)$ which fails to be well-defined in the polymer Hilbert space $\mathcal{H}_{\rm poly}$.

ii) From the strict perspective of Minkowskian field theories, the presence of the taming function $f_{r,x}(y)$ (and especially its new scale r) in all these expressions seems awkward. From the quantum geometry perspective, this is because the 'physical origin' of this function lies beyond the scope of the continuum theories. The viewpoint is that, in Nature, there is fundamental discreteness because of the quantum nature of geometry and the continuum geometry appears only on suitable coarse graining of the fundamental, quantum geometry. It is this coarse graining that provides the taming function. In effect, the discrete geometry can be approximated by the continuum only if we define 'effective geometric fields' at each point x in the continuum by averaging, i.e., by smearing the fundamental quantum-geometric structures with a function $f_{r,x}(y)$. On general grounds, one expects the discreteness to appear at the Planck scale ℓ_P. Hence the averaging length scale r has to be much larger than ℓ_P but much smaller than the length scales probed by the energies available in given experiments. Thus, as one 'descends from the fundamental, Planck scale perspective', one will be led to an effective description involving $f_{r,x}$ —as manifested, e.g., in the r-Fock representation. But one can then notice that by 'undoing the taming map' one can get rid of $f_{r,x}$ provided one works with the space $\mathcal{S}^\star$ of distributions. From this perspective, then, r-Fock representation is an intermediate step; it is awkward because it has neither the mathematical elegance that working in the continuum from the beginning brings nor the deeper physical insights that working with quantum geometry can bring.

IV. COMPARISON

Let us begin with algebras. The standard Weyl algebra can be obtained by a suitable completion of the algebra generated by the operators $\hat{U}(\lambda f_{r,x})$ and $\hat{V}(g) := \exp i\hat{\pi}(g)$. These operators satisfy the well-known commutation relations

$$\hat{V}(g)\,\hat{U}(\lambda f_{r,x}) = e^{i\hbar\lambda\int d^3y f_{r,x}(y)g(y)}\ \hat{U}(\lambda f_{r,x})\,\hat{V}(g). \tag{4.1}$$

On the polymer side, consider operators $\hat{h}(x,\lambda)$ and $\hat{\mathcal{V}}(f_r \star g) = \exp i\hat{P}(f_{r,x} \star g)$, where $(f_r \star g)(y) = \int d^3x f_r(x,y)g(x)$ is the convolution of f_r and g. Their commutation relations are

$$\hat{\mathcal{V}}(f_r \star g)\hat{h}(x,\lambda) = e^{i\hbar\lambda\int d^3y f_{r,x}(y)g(y)}\ \hat{h}(x,\lambda)\hat{\mathcal{V}}(f_r \star g) \tag{4.2}$$

Hence,

$$(\hat{U}(\lambda f_{r,x}),\, \hat{V}(g)) \longmapsto (\hat{h}(x,\lambda),\, \hat{\mathcal{V}}(f_r \star g))$$

defines a *-isomorphism between the two algebras. Put differently, in this letter we have discussed two inequivalent representations of the standard Weyl algebra; the Fock and the Polymer.

Next, let us now compare the polymer and the r-Fock representations. Both Hilbert spaces, $\mathcal{H}_{\rm poly}$ and $\mathcal{H}^{(r)}_{\rm Fock}$, can be obtained by Cauchy completing Cyl but using inner products determined by the respective measures $d\mu_o$ and $d\mu^{(r)}_F$. On Cyl, the holonomy operators of the polymer representation and the configuration operators of the r-Fock representation are related by (see (2.3) and (3.6)):

$$[(\hat{h}_{\lambda_o,x_o} - \hat{U}_{(r)}(\lambda_o f_{r,x_o})]\,\Psi = 0$$

for all cylindrical functions Ψ on $\bar{\mathcal{A}}$. Next, let us consider the momentum operators. In the polymer representation we have (see (2.6))

$$[\hat{P}(g),\, \hat{h}(x_o,\lambda_o)] \,=\, \hbar\lambda_o\, g(x_o)\,\hat{h}(x_o,\lambda_o)$$

while in the r-Fock representation we have (see (3.8))

$$[\hat{\pi}_{(r)}(g),\, \hat{U}_r(\lambda_o f_{r,x_o})] = \hbar\lambda_o\,[\int\, d^3x g(x) f_{r,x_o}(x)]\,\hat{U}_r(\lambda_o f_{r,x_o})]$$

Therefore, the appropriate operators to compare are

$$\hat{P}(g \star f_r) \quad \text{and} \quad \hat{\pi}_{(r)}(g)\,.$$

In the action of these operators on Cyl, the first term ('Lie derivative along the vector fields') is the same. But while the vector fields on $\bar{\mathcal{A}}$ defining these momenta are divergence free with respect to μ_o, they are not with respect to $\mu^{(r)}_F$. Hence in the expression of $\hat{\pi}_r(x)$ there is an extra term. It is this difference that makes the two representations of the Weyl algebra unitarily inequivalent.

Finally, let us compare the two measures. For this, we can calculate the integrals of general elements of Cyl with respect to both. Given a vertex set $V = \{x_1, \ldots x_n\}$, let us define

$$G_{ij} = \int d^3y \; f_{r,x_i}(y)\, \Delta^{-\frac{1}{2}} f_{r,x_j}(y)\,.$$

For basis functions $\mathcal{N}_{V,\vec{\lambda^o}}(\phi)$ associated with V, we have:

$$\begin{aligned}
&\int_{\bar{\mathcal{A}}} d\mu_o \,[\sum_{\vec{\lambda}} \; e^{-\frac{1}{4}\sum_{j,k} G_{jk}\lambda_j\lambda_k} \bar{\mathcal{N}}_{V,\vec{\lambda}}(\phi)]\; \mathcal{N}_{V,\vec{\lambda_o}}(\phi) \\
&= \sum_{\vec{\lambda}} \; e^{-\frac{1}{4}\sum_{j,k} G_{jk}\lambda_j\lambda_k} \; \int_{\bar{\mathcal{A}}} d\mu_o \; \overline{\mathcal{N}}_{V,\vec{\lambda}}(\phi)\; \mathcal{N}_{V,\vec{\lambda_o}}(\phi) \\
&= e^{-\frac{1}{4}\sum_{ij} G_{ij}\,\lambda_i^o\lambda_j^o} \\
&= \int_{\bar{\mathcal{A}}} d\mu_F^{(r)}\; \mathcal{N}_{V,\vec{\lambda_o}}(\phi)
\end{aligned} \tag{4.3}$$

Therefore, we can write the relation between the two measures on $\bar{\mathcal{A}}$ as

$$d\mu_F^{(r)}(\phi) = [\sum_{V,\vec{\lambda}} \; e^{-\frac{1}{4}\sum_{j,k} G_{jk}\lambda_j\lambda_k} \overline{\mathcal{N}}_{V,\vec{\lambda}}(\phi)\,]\; d\mu_o(\phi)\,. \tag{4.4}$$

Since the sum is over *continuous* variables $(V, \vec{\lambda})$, the quantity in square brackets is *not* a function on $\bar{\mathcal{A}}$. In fact, using the results of [11], one can show that the two measures are inequivalent.[2] Thus, the equation is to be understood only in the sense of distributions: every element of Cyl is integrable with respect to both sides and the value of the integral with respect to the measure on the right side equals that on the left.

Let us summarize. The $C^\star$ algebra generated by $\hat{U}(\lambda f_{r,x})$ and $\hat{V}(g)$ —i.e., the standard Weyl algebra —admits two unitarily inequivalent representations. In both representations a dense subset of states is provided by Cyl; the quantum configuration space $\bar{\mathcal{A}}$ provides a 'common home'. However, the completions are with respect to inequivalent measures, $d\mu_o$ and $d\mu_F^{(r)}$ on $\bar{\mathcal{A}}$. The configuration operators act by multiplication on Cyl in both cases. But the action of the momentum operators differs. In the polymer representation, the unitary operators $\hat{U}(\lambda f_{r,x})$ fail to be weakly continuous in λ whence $\hat{\phi}(x)$ fail to exist as operator-valued distributions while in the r-Fock representations the weak continuity holds and the operator-valued distributions exist. The polymer representation makes no reference to a metric on R^3; all constructions are covariant with respect to the diffeomorphism group on R^3. The r-Fock representation, of course, is tied to the flat metric.

Remark: In the Maxwell case, the λ_j are replaced by integers n_j and the vertex sets V by graphs γ. Therefore, when we restrict ourselves to a fixed graph γ, in the analog of (4.4) the only sum involved is over integers. The restriction $d\mu_{F,\gamma}^{(r)}$ of the r-Fock measure and the restriction $d\mu_{o,\gamma}$ of the polymer measure to the graph γ are related by a continuous function, whence these two measures are *absolutely continuous* with respect to one another [2]. In the

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[2]In particular, there is *no* state in $\mathcal{H}_{\rm poly}$ for which the expectation value of the holonomy $\exp i[\lambda_o\phi(x_o)]$ equals $\int_{\bar{\mathcal{A}}} d\mu_F^{(r)} \exp i[\lambda_o\phi(x_o)]$ for all pairs x_o, λ_o.

scalar field case, since λ_j are continuous labels, even the restrictions of the two measures to any one vertex set V fail to be absolutely continuous. To obtain absolute continuity, one has to further restrict the λ_js to a *countable* set. Consequently, the discussion of *shadow states* [2] now requires this additional restriction.

V. OUTLOOK

Introduction of the polymer representation raises two obvious questions: i) We know that the Fock representation can be used very effectively to describe low energy physics. How would this description arise from the 'fundamental' theory which is based on quantum geometry and polymer description of quantum fields?; and, ii) Can the 'fundamental' framework address any of the open problems of quantum field theory?

A comprehensive answer to these questions would require a step by step procedure which starts from the solutions to the quantum constraints in the coupled gravity and matter theory and analyzes them in detail in the semi-classical sector of the theory. Such a detailed reduction is yet to be constructed. So, we will adopt an optimistic viewpoint, assume that the missing intermediate steps can be filled, and summarize the final picture envisaged today.

The semi-classical state of quantum geometry corresponding to Minkowski space will provide a graph γ and a quantum geometry state on it (more precisely [2], an element of $\mathrm{Cyl}^\star$ of quantum geometry which assigns amplitudes to each state defined on a *suitable* family of graphs γ). Roughly, the edge lengths and the average separation between vertices of these graphs would be a Planck length ℓ_P (as measured by the continuum geometry at which the semi-classical state is peaked). The flat continuum geometry would arise only when we coarse grain the state with a smoothening function $f_{r,x}$ with $\ell_P \ll r \ll \hbar/E$ where E is the highest energy scale we are interested in. This smoothening function will be then used also in constructing the r-Fock representations of matter fields. In what follows, for simplicity we will refer to quantum field theories on given quantum geometries as 'fundamental' and regard the coarse grained continuum description as an approximation.

Let us return to the scalar field. The 'fundamental' scalar field will reside only on the vertex set V of each of these graphs γ. Thus, qualitatively, we have a lattice field theory. If we only consider states in Cyl_V, the restricted polymer and r-Fock descriptions will be unitarily equivalent (once the restriction mentioned in the remark at the end of section IV is made). Basically, this is an illustration of Fell's theorem [12]: from the continuum perspective we are looking at a restricted class of observables in both theories. But the viewpoint is that *this restricted framework is a more fundamental description than the continuum one.* Recall that predictions of renormalizable theories for energy scales E are insensitive to the details of the structure at length scales $L \ll \hbar/E$. Therefore, for such theories, the restricted framework will provide a (generalized) lattice description whose predictions are borne out in the low energy experiments. Observationally, these predictions will be indistinguishable from those of the continuum Fock theory. From the 'fundamental' point of view, these predictions are really derived from the polymer description, restricted to the graphs selected by the semi-classical state of the quantum geometry and the remark about agreement with the r-Fock description is a only a quick way to check that these results are observationally viable. Thus, the viewpoint is that neither the r-Fock nor the polymer description in the continuum

is fundamental. However, the continuum Fock description is a very useful approximation while the continuum polymer scalar field description is not likely to be directly useful in the familiar situations.[3] The real arena for the polymer description is quantum geometry.

The 'fundamental' description should be able to shed new light on quantum field theory issues. We will conclude with some examples. First, all results of the 'fundamental theory' are expected to be finite since by construction the theory would be free of ultraviolet divergences.[4] Therefore it should be possible to trace the precise manner in which the continuum approximation leads to these divergences. Second, it may provide a new physical basis, rooted in quantum geometry, for the Wilsonian ideas of renormalization flows. As we increase the energy scale E (still keeping it well below the Planck scale) we have to use smaller r and more refined graphs and one can study how physical results transform with these refinements. Third, since the smoothening procedure introduces a scale (r), it may account for the known emergence of new scales in quantum field theories (e.g. of zero rest mass fields) which are absent in the classical theory. Finally, the smoothening procedure also introduces a small and subtle degree of non-locality which could play an important conceptual role.

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[3]For *gauge fields*, the continuum polymer description may be useful, e.g., in describing type II superconductors where flux of the magnetic field is quantized; or, in understanding some issues related to confinement in QCD.

[4]Note that, although many of the techniques used are the same as in lattice gauge theory, the viewpoint here is diametrically opposite. Here (generalized) lattices are not convenient mathematical constructs to approximate the continuum theories; they are provided by the physical quantum geometry and therefore *more* fundamental than the continuum. We can and do work with the continuum limit but primarily for mathematical convenience, e.g., because differential equations and integrals are often easier to control than difference equations and discrete sums.

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